

BASIC MORSE-NOVIKOV COHOMOLOGY FOR FOLIATIONS

LIVIU ORNEA AND VLADIMIR SLESAR

ABSTRACT. In this paper we find sufficient conditions for the vanishing of the Morse-Novikov cohomology on Riemannian foliations. We work out a Bochner technique for twisted cohomological complexes, obtaining corresponding vanishing results. Also, we generalize for our setting vanishing results from the case of closed Riemannian manifolds. Several examples are presented, along with applications in the context of l.c.s. and l.c.K. foliations.

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1. INTRODUCTION

We consider in what follows a smooth manifold M endowed with a global closed differential one-form θ . The *Morse-Novikov cohomology complex* (Ω, d_θ) (where Ω is the de Rham complex of the manifold M , while d_θ is the *twisted derivative* $d_\theta := d - \theta \wedge$) plays an important role when investigating aspects related to the geometry, topology and Morse theory of the underlying manifold M (see *e.g.* [Pa]).

Cohomological complexes of this type were also introduced and studied by Lichnerowicz in the context of Poisson geometry [L] (in many papers this cohomology is also called *Lichnerowicz cohomology*).

Classical examples of Morse-Novikov cohomology are obtained on locally conformally symplectic manifolds and locally conformally Kähler manifolds [OV2, V3]. These manifolds admit local symplectic and Kähler structures which cannot be extended to the whole manifold. Instead, at the global level a closed one-form is obtained (called the *Lee form*), and a Morse-Novikov cohomology naturally appears.

Twisted differential operators and Morse-Novikov cohomology can be canonically extended in the larger framework represented by Riemannian foliations (*i.e.* foliations with Riemannian structure that locally induces Riemannian submersions [R]). The transverse geometry of the foliations represents the extension of the geometry of Riemannian manifolds; the classical setting is obtained in the absolute case of manifolds foliated by points [Mo, T].

On Riemannian foliations defined on a closed manifold, such twisted basic cohomological objects are mostly used for the case when θ is related to the *mean curvature form*. For example, in [Do] the author studies the tenseness of Riemannian foliations (*i.e.* the existence of a metric with basic (projectable) mean curvature form). In [HR], twisted (modified) differentials are used as an alternative to the

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classical approach in order to investigate the tautness of the foliation (*i.e.* the existence of a metric which turns all leaves into minimal submanifolds), and to perform basic harmonic analysis.

In this paper we stick with the general case of basic Morse-Novikov cohomology associated to a basic, closed one-form, and present several instances in which the groups of this cohomology are trivial.

First of all, we generalize to our setting previous vanishing results known in the classical case when θ is parallel [LLMP] and, respectively, non-exact [GL]. Then we study the influence of the basic curvature on the basic Morse-Novikov cohomological groups. In addition to the case when this curvature operator is non-negative, in the second part of the paper we work out a Bochner-type technique specific for our particular framework.

Consequently, we obtain vanishing results which may hold even in the case when the basic curvature is not necessarily non-negative (in this case the basic de Rham cohomology groups may not be trivial).

Concerning the last result, two particular cases are relevant. These are the case of classical closed Riemannian manifolds and the case of the basic de Rham cohomology of Riemannian foliations. On the other hand, the above vanishing results have analogues in the context of locally conformally symplectic and locally conformally Kähler foliations.

The paper is organized as follows. In the next section we present the main features of Riemannian foliations with basic mean curvature form, which represent our framework throughout this paper. In section 3 we introduce several technical tools we use in the rest of the paper. More precisely, the three subsections present the twisted Bott connection, the twisted basic curvature operator and a corresponding Weitzenböck formula. In section 4 we present the main results of the paper along with several examples. The consequences of these results for the setting of locally conformally symplectic and locally conformally Kähler foliations are briefly stated in the final section.

2. PRELIMINARIES

2.1. Basic facts about foliations. We consider in what follows a smooth, closed Riemannian manifold (M, g, \mathcal{F}) endowed with a foliation \mathcal{F} such that the metric g is bundle-like [R]; the dimension of M will be denoted by n . We denote by $T\mathcal{F}$ the leafwise distribution tangent to leaves. A classical vector bundle constructed on M is $Q := TM/T\mathcal{F}$ (see *e.g.* [T]). Note that the definition of Q does not require the metric g . Considering g , we can further obtain $Q \simeq T\mathcal{F}^\perp$. For convenience, in the following we denote also the transverse distribution $T\mathcal{F}^\perp$ by Q , in accordance with [A]. Assume $\dim T\mathcal{F} = p$, $\dim Q = q$, so $p + q = n$.

As a consequence, we get the following exact sequence of vector bundles

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \longrightarrow Q \longrightarrow 0.$$

A corresponding exact sequence for the dual vector bundles also appears (see *e.g.* [T]). The canonical projection operators on the distributions Q and $T\mathcal{F}$ will be denoted by π_Q and $\pi_{T\mathcal{F}}$, respectively.

Throughout this paper we use local vector fields $\{e_i, f_a\}$ defined on a neighborhood of an arbitrary point $x \in M$, so that they determine an orthonormal basis at any point where they are defined, $\{e_i\}$ spanning the distribution Q and $\{f_a\}$

spanning the distribution $T\mathcal{F}$. In what follows we use the classical ‘musical’ isomorphisms \sharp and \flat determined by the metric structure g . The coframe $\{e^i, f^a\}$ will be also employed, with $e^i := e_i^\flat$, $f^a := f_a^\flat$.

A standard linear connection used in the study of the basic geometry of our Riemannian foliated manifold is the *Bott connection* (see e.g. [T]); it is a metric and torsion-free connection traditionally defined on the smooth sections of the quotient bundle $TM/T\mathcal{F}$. According to our above considerations, we will define the Bott connection ∇ on the sections of Q by the following relations

$$\begin{cases} \nabla_u w := \pi_Q([u, w]), \\ \nabla_v w := \pi_Q(\nabla_v^M w), \end{cases}$$

for any smooth section $u \in \Gamma(T\mathcal{F})$ and $v, w \in \Gamma(Q)$.

The transverse divergence associated to the Bott connection is defined in the usual manner, as a trace operator :

$$\operatorname{div}^\nabla := \sum_i g(\nabla_{e_i} \cdot, e_i).$$

As in the case of Riemannian submersions, we investigate the geometric objects that can be locally projected on submanifolds transverse to the leaves. The restriction of the classical de Rham complex of differential forms $\Omega(M)$ to the complex of basic (projectable) differential forms generates the *basic de Rham complex*, defined as

$$\Omega_b(\mathcal{F}) := \{\eta \in \Omega(M) \mid \iota_v \eta = 0, \mathcal{L}_v \eta = 0 \text{ for any } v \in \Gamma(T\mathcal{F})\}.$$

Here \mathcal{L} is the Lie derivative along v , while ι stands for interior product. The *basic de Rham derivative* is defined also as a restriction of the classical derivative d , namely $d_b := d|_{\Omega_b(\mathcal{F})}$ (see e.g. [T]).

Remark 2.1. The basic de Rham complex is defined independently of the metric structure g , and in fact the groups of the basic de Rham cohomology are topological invariants [T].

Remark 2.2. A direct computation shows that basic forms are parallel along the leaves with respect to the connection ∇ associated to a bundle-like metric. A *basic vector field* is a vector field v parallel along leaves with respect to ∇ . If the vector field is also transverse, i.e. it is a section in the transverse distribution, $v \in \Gamma(Q)$, then $v^\flat \in \Omega_b(\mathcal{F})$. In the sequel, we use basic (projectable) vector fields $\{e_i\}$.

For any basic forms $\alpha_1, \alpha_2 \in \Omega_b^p(\mathcal{F})$, we extend the notation $g(\alpha_1, \alpha_2)$ for the inner product canonically induced by the metric tensor g on $\Omega_b^p(\mathcal{F})$. Taking the integral on the closed manifold M , we obtain the classical L^2 inner product

$$\langle \alpha_1, \alpha_2 \rangle := \int_M g(\alpha_1, \alpha_2) d\mu_g,$$

where $d\mu_g$ is the measure induced on M by g .

An interesting example of differential form which is not necessarily basic is represented by the *mean curvature form* (see e.g. [A, BEI]). It is denoted by κ and it is defined as

$$\kappa^\sharp := \pi_Q\left(\sum_a \nabla_{f_a}^M f_a\right).$$

According to [A], on any Riemannian foliation the mean curvature form can be decomposed in a unique way as the sum

$$\kappa = \kappa_b + \kappa_o,$$

where κ_b is the L^2 -orthogonal projection of κ onto the closure of $\Omega_b(\mathcal{F})$, with κ_o being its orthogonal complement. We note that κ_b is smooth and closed [A].

A fundamental result in this field is due to Domínguez [Do], and it shows that κ_o can always be considered to be 0. More precisely, any Riemannian foliation defined on a closed manifold can be turned into a foliation with basic mean curvature form by changing the bundle-like metric such that the transverse metric remains unchanged. As we plan to work with geometric objects related to the transverse metric structure, we can make the standard assumption of a basic mean curvature without actually restricting our framework (see *e.g.* [HR, T]).

And hence, from now on we shall assume that

$$\boxed{\kappa = \kappa_b}$$

We canonically extend the Bott connection ∇ on $\Omega_b(\mathcal{F})$; this extension is denoted by ∇ as well. The adjoint operator of d_b with respect to the above hilbertian product, which is called the *basic de Rham coderivative*, may be also written as [A]

$$\delta_b := \sum_i -\iota_{e_i} \nabla_{e_i} + \iota_{\kappa^\#}.$$

2.2. Morse-Novikov cohomology with basic form. On a closed Riemannian manifold M , using a closed differential 1-form θ , we can define the twisted de Rham derivative $d_\theta : \Omega(M) \rightarrow \Omega(M)$,

$$d_\theta := d - \theta \wedge,$$

where $\Omega(M)$ is the de Rham complex defined on M .

As θ is closed, $d_\theta^2 = 0$, and the Morse-Novikov cohomological complex $(\Omega(M), d_\theta)$ can be defined canonically. Note that if θ is exact, $\theta = df$, $f \in \mathcal{C}^\infty(M)$, then the mapping $[\alpha] \rightarrow [e^{-f}\alpha]$ defines an isomorphism between the de Rham and Morse-Novikov cohomologies.

Also, the concept of Morse-Novikov cohomology can be easily extended to the basic Morse-Novikov cohomology on a Riemannian foliation. More precisely, assuming that θ is a basic closed one-form, then we can write the *twisted basic de Rham derivative* using the above defined basic de Rham differential operator d_b (see *e.g.* [IP])

$$d_{b,\theta} := d_b - \theta \wedge,$$

and the basic Morse-Novikov complex $(\Omega_b(\mathcal{F}), d_{b,\theta})$ is constructed. The Morse-Novikov cohomology groups $\{H_{b,\theta}^i(\mathcal{F})\}_{0 \leq i \leq q}$ (which can be regarded as twisted de Rham cohomology groups) are defined in the usual way.

The particular case $\theta = \frac{1}{2}\kappa$ is investigated in [HR], the authors obtaining vanishing results using curvature-type operators and an interesting interplay with the tautness properties of the foliation.

For general θ , in order to study these cohomological groups, we define basic twisted differential operators compatible with $d_{b,\theta}$.

We introduce these operators in the next section.

3. BASIC TWISTED DIFFERENTIAL OPERATORS ON RIEMANNIAN FOLIATIONS

3.1. The twisted Bott connection. The main tool we use to describe and investigate the twisted cohomology groups $H_{b,\theta}^i(\mathcal{F})$ is a linear connection that we modify in a convenient way.

Definition 3.1. For a closed one-form $\theta \in \Omega_b(\mathcal{F})$ and vector fields $v \in \Gamma(TM)$, $w \in \Gamma(Q)$ we define the *twisted Bott connection* ∇^θ

$$\nabla_v^\theta w := \nabla_v w - \theta(v)w.$$

For the particular case when $\theta = \frac{1}{2}\kappa$ we adopt the notation $\nabla^{\frac{1}{2}\kappa} := \tilde{\nabla}$. As this connection will play an important role in our further considerations, we choose to denote

$$(1) \quad \tilde{\nabla}^\theta := \nabla^{\frac{1}{2}\kappa + \theta}.$$

The connection in (1) is extended canonically to $\Omega_b(\mathcal{F})$, and for convenience it will be denoted as $\tilde{\nabla}^\theta$, too.

Remark 3.1. As $\kappa^\sharp, \theta^\sharp \in \Gamma(Q)$, if $\alpha \in \Omega_b(\mathcal{F})$ and $v \in \Gamma(T\mathcal{F})$, then

$$\tilde{\nabla}_v^\theta \alpha = 0,$$

in other words, *the basic forms are ‘leafwise’ parallel with respect to the new connection $\tilde{\nabla}^\theta$.*

For the complementary case we have the following result.

Lemma 3.1. *If v is a basic vector field, then the operators ∇_v and $\tilde{\nabla}_v^\theta$ map basic forms to basic forms.*

Proof. We first prove the statement for ∇ and basic one-forms. Let w be a basic and transverse vector field. As, locally, a Riemannian foliation can be identified with a Riemannian submersion, from [ON, Lemma 1.(3)] and the definition of ∇ we see that $\nabla_v w$ is basic and transverse. By Remark 2.2, one-forms can be related to basic and transverse vector fields using the musical isomorphism induced by the bundle-like metric g . As the connection ∇ is metric, ∇_v maps basic one-forms to basic one-forms. We use the local coframe $\{e^i, f_a\}$ with $\{e^i\}$ basic forms. Any basic form α of dimension u , $1 \leq u \leq p$, can be written

$$\alpha = \sum_{1 \leq i_1 < \dots < i_u \leq p} f_{i_1, \dots, i_u} e^{i_1} \wedge \dots \wedge e^{i_u},$$

with smooth functions f_{i_1, \dots, i_u} . We obtain

$$\begin{aligned} \nabla_v \alpha &= \sum_{1 \leq i_1 < \dots < i_u \leq p} v(f_{i_1, \dots, i_u}) e^{i_1} \wedge \dots \wedge e^{i_u} \\ &+ \sum_{1 \leq i_1 < \dots < i_u \leq p} f_{i_1, \dots, i_u} \nabla_v e^{i_1} \wedge \dots \wedge e^{i_u} + \dots \\ &+ \sum_{1 \leq i_1 < \dots < i_u \leq p} f_{i_1, \dots, i_u} e^{i_1} \wedge \dots \wedge \nabla_v e^{i_u}. \end{aligned}$$

Using [Mo, Proposition 2.2], the functions $v(f_{i_1, \dots, i_u})$ are basic. Then ∇_v maps basic forms to basic forms.

As for $\tilde{\nabla}_v^\theta$, note that $\theta, \kappa \in \Omega_b(\mathcal{F})$ and v are basic, and hence $(\frac{1}{2}\kappa + \theta)(v)$ is a basic function (constant on the leaves), so that the connection $\tilde{\nabla}^\theta$ maps basic forms to basic forms. \square

The interesting feature of the twisted Bott connection $\tilde{\nabla}^\theta$ is that it can be used to build up the twisted basic de Rham derivative. Denote [IP]

$$\tilde{d}_{b,\theta} := d_{b,\frac{1}{2}\kappa+\theta}.$$

An alternative way to construct this operator is the following:

$$(2) \quad \tilde{d}_{b,\theta} = \sum_i e^i \wedge \tilde{\nabla}_{e_i}^\theta.$$

We will also denote the cohomology groups associated to $\tilde{d}_{b,\theta}$ by $\tilde{H}_{b,\theta}^i(\mathcal{F})$, with $\tilde{H}_{b,\theta}^i(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^i(\mathcal{F})$.

For the particular case $\theta \equiv 0$, one reobtains the basic modified operator $\tilde{d}_b := \tilde{d}_{b,0}$, with $\tilde{d}_{b,\theta} = \tilde{d}_b - \theta \wedge$, and the corresponding cohomology complex $\tilde{H}_b^i(\mathcal{F}) := \tilde{H}_{b,0}^i(\mathcal{F})$, in accordance with [HR].

3.2. Computational properties of the twisted basic operators. In the following we investigate several computational properties of the above introduced twisted basic operators (again, for the case $\theta \equiv 0$ see also [HR, ISVV]).

Consider a basic vector field v . Using Lemma 3.1, we see that the operators $\tilde{\nabla}^\theta$ and $\tilde{d}_{b,\theta}$ send basic forms to basic forms. In the following we construct the *basic adjoint operators* $\tilde{\nabla}_v^{\theta*}$ and $\tilde{\delta}_{b,\theta} := \tilde{d}_{b,\theta}^*$, i.e. the adjoint operators with respect to the restriction of the above L^2 inner product to the complex of basic forms; in the remaining part of the paper each time by adjoint operators we will understand basic adjoint operators. For the twisted Bott connection we obtain the following result.

Lemma 3.2. *The (formal) adjoint operator associated to the differential operator $\tilde{\nabla}_v^\theta$ can be computed with the formula*

$$(3) \quad \tilde{\nabla}_v^{\theta*} = -\tilde{\nabla}_v^\theta - \operatorname{div}^\nabla v - 2\theta(v).$$

Proof. Consider the operator $T := -\tilde{\nabla}_v^\theta - \operatorname{div}^\nabla v - 2\theta(v)$. For all smooth basic forms α_1 and α_2 we evaluate the expression

$$S := \langle \tilde{\nabla}_v^\theta \alpha_1, \alpha_2 \rangle - \langle \alpha_1, T \alpha_2 \rangle.$$

We obtain

$$(4) \quad \begin{aligned} \langle \tilde{\nabla}_v^\theta \alpha_1, \alpha_2 \rangle &= \langle \nabla_v \alpha_1, \alpha_2 \rangle - \frac{1}{2} \langle \kappa(v) \alpha_1, \alpha_2 \rangle \\ &\quad - \langle \theta(v) \alpha_1, \alpha_2 \rangle \end{aligned}$$

and

$$(5) \quad \begin{aligned} \langle \alpha_1, T \alpha_2 \rangle &= -\langle \alpha_1, \nabla_v \alpha_2 \rangle + \frac{1}{2} \langle \alpha_1, \kappa(v) \alpha_2 \rangle \\ &\quad - \langle \alpha_1, \theta(v) \alpha_2 \rangle - \langle \alpha_1, (\operatorname{div}^\nabla v) \alpha_2 \rangle. \end{aligned}$$

From (4) and (5), similarly to the classical case, we get

$$\begin{aligned}
S &= \int_M v(g(\alpha_1, \alpha_2)) d\mu_g + \int_M \operatorname{div}^\nabla v g(\alpha_1, \alpha_2) d\mu_g \\
&\quad + \int_M g(\nabla_{f_a}^M v, f_a) g(\alpha_1, \alpha_2) d\mu_g \\
&= \int_M (v(g(\alpha_1, \alpha_2)) + \operatorname{div} v g(\alpha_1, \alpha_2)) d\mu_g \\
&= \int_M \operatorname{div} (g(\alpha_1, \alpha_2) v) = 0 \quad \text{by Green theorem, [Po]}.
\end{aligned}$$

Then, T is in fact the formal adjoint operator of $\tilde{\nabla}_v^\theta$, so

$$\tilde{\nabla}_v^{\theta*} = -\tilde{\nabla}_v^\theta - \operatorname{div}^\nabla v - 2\theta(v).$$

□

We now compute the adjoint operator $\tilde{\delta}_{b,\theta} = \tilde{d}_{b,\theta}^*$. Consider first the operator

$$\begin{aligned}
\tilde{\delta}_b &:= \sum_i -\iota_{e_i} \nabla_{e_i} + \frac{1}{2} \iota_{\kappa^\sharp} \\
&= \delta_b - \frac{1}{2} \iota_{\kappa^\sharp}
\end{aligned}$$

which is known to be the adjoint of \tilde{d}_b , [HR]. Then, the operator

$$\begin{aligned}
\tilde{\delta}_{b,\theta} &:= \tilde{\delta}_b - \iota_{\theta^\sharp} \\
(6) \quad &= \sum_i -\iota_{e_i} \tilde{\nabla}_{e_i}^\theta - 2\iota_{\theta^\sharp}
\end{aligned}$$

is the adjoint of $\tilde{d}_{b,\theta}$.

Let v and w be transverse basic vector fields and let α be a basic differential form. We define the Clifford product

$$v \cdot \alpha := v^\flat \wedge \alpha - \iota_v \alpha.$$

We remark that $v \cdot v \cdot \alpha = -\|v\|_g^2 \alpha$, where $\|v\|_g := \sqrt{g(v, v)}$.

We define the corresponding Dirac-type operator $\tilde{D}_{b,\theta}$ using (2) and (6)

$$\begin{aligned}
\tilde{D}_{b,\theta} &:= \tilde{d}_{b,\theta} + \tilde{\delta}_{b,\theta} \\
&= \sum_i (e^i \wedge \tilde{\nabla}_{e_i}^\theta - \iota_{e_i} \tilde{\nabla}_{e_i}^\theta) - 2\iota_{\theta^\sharp} \\
&= \sum_i e_i \cdot \tilde{\nabla}_{e_i}^\theta - 2\iota_{\theta^\sharp}.
\end{aligned}$$

As in the classical case, a Laplace-type operator related to $\tilde{d}_{b,\theta}$ can be defined

$$\tilde{\Delta}_{b,\theta} := \tilde{d}_{b,\theta} \tilde{\delta}_{b,\theta} + \tilde{\delta}_{b,\theta} \tilde{d}_{b,\theta}.$$

Remark 3.2. $\tilde{\Delta}_{b,\theta}$ is a transverse elliptic operator defined on the Riemannian foliation, with the same symbol as Δ_b .

For $\theta = 0$, we obtain the twisted operator $\tilde{\Delta}_b$ employed in [HR]; furthermore, for $\theta = -\frac{1}{2}\kappa$ we actually obtain the basic Laplace operator Δ_b (see *e.g.* [RP, T]).

$(\Omega_b(\mathcal{F}), \tilde{d}_{b,\theta})$ is a transverse elliptic complex and hence, similarly to [HR, Proposition 2.3] (for the classical case when the manifold is foliated by points see *e.g.* [Gi, V2]), the following Hodge-type decomposition holds:

Theorem 3.1. ([IP]) *The basic cohomology $\Omega_b(\mathcal{F})$ can be written as a direct sum*

$$\Omega_b(\mathcal{F}) = \text{Im}(\tilde{d}_{b,\theta}) \oplus \text{Im}(\tilde{\delta}_{b,\theta}) \oplus \text{Ker}(\tilde{\Delta}_{b,\theta}).$$

If we denote $\mathcal{H}^p(\tilde{\Delta}_{b,\theta}) := \text{Ker} \tilde{\Delta}_{b,\theta}|_{\Omega_b(\mathcal{F})}$, with $0 \leq p \leq n$, then

$$\mathcal{H}^p(\Delta_\theta) \simeq \tilde{H}_{b,\theta}^p(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^p(\mathcal{F}).$$

Now, concerning the twisted Bott connection and the Clifford product, we have

Lemma 3.3. *The following Leibniz rule holds:*

$$\tilde{\nabla}_v^\theta(w \cdot \alpha) = \nabla_v w \cdot \alpha + w \cdot \tilde{\nabla}_v^\theta \alpha.$$

Proof. We have

$$\begin{aligned} \tilde{\nabla}_v^\theta(w \cdot \alpha) &= \nabla_v(w \cdot \alpha) - \left(\frac{1}{2}\kappa + \theta\right)(v)w \cdot \alpha \\ &= \nabla_v w \cdot \alpha + w \cdot (\nabla_v \alpha - \left(\frac{1}{2}\kappa + \theta\right)(v)\alpha), \end{aligned}$$

and the result follows from the very definition of $\tilde{\nabla}_v^\theta$. \square

Lemma 3.4. *The following relation holds:*

$$\iota_w \tilde{\nabla}_v^\theta = \tilde{\nabla}_v^\theta \iota_w - \iota_{\nabla_v w}.$$

Proof. We can write

$$\begin{aligned} \iota_w \tilde{\nabla}_v^\theta &= \iota_w (\nabla_v - \left(\frac{1}{2}\kappa + \theta\right)(v)) \\ &= \nabla_v \iota_w - \iota_{\nabla_v w} - \left(\frac{1}{2}\kappa + \theta\right)(v) \iota_w \\ &= \tilde{\nabla}_v^\theta \iota_w - \iota_{\nabla_v w}. \end{aligned}$$

\square

The following two equations relating standard operators on Riemannian foliations are the natural extension of classical results from calculus on differentiable manifolds. The proofs are similar to the classical case.

Lemma 3.5. *For any basic vector fields v, w and basic form $\alpha \in \Omega_b(\mathcal{F})$, we have*

$$\begin{aligned} \iota_w(v \cdot \alpha) &= \iota_w v^\flat \alpha - v \cdot \iota_w \alpha, \\ (7) \quad \mathcal{L}_v \alpha - \nabla_v \alpha &= \sum_i e^i \wedge \iota_{\nabla_{e_i} v} \alpha. \end{aligned}$$

3.3. The twisted basic curvature operator. In this subsection we show that the curvature operator associated to the twisted Bott connection coincides in fact with the basic curvature operator and depends only on the transverse metric.

Let γ be a closed basic one-form and denote by $R_{v,w}^\gamma$ the basic curvature operator written using the connection ∇^γ and the transverse basic vector fields v, w . We have the following useful relation:

Lemma 3.6. *The curvature operator $R_{v,w}^\gamma$ does not depend on γ , namely*

$$R_{v,w}^\gamma = R_{v,w}.$$

Proof. Starting with the definition of the curvature operator, we obtain for $R_{v,w}^\gamma$

$$(8) \quad R_{v,w}^\gamma = \nabla_v^\gamma \nabla_w^\gamma - \nabla_w^\gamma \nabla_v^\gamma - \nabla_{[v,w]}^\gamma.$$

Furthermore, we compute:

$$\begin{aligned} \nabla_v^\gamma \nabla_w^\gamma &= (\nabla_v - \gamma(v))(\nabla_w - \gamma(w)) \\ &= \nabla_v \nabla_w - \gamma(v) \nabla_w - v(\gamma(w)) \\ &\quad - \gamma(w) \nabla(v) + \gamma(v) \gamma(w). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \nabla_w^\gamma \nabla_v^\gamma &= (\nabla_w - \gamma(w))(\nabla_v - \gamma(v)) \\ &= \nabla_w \nabla_v - \gamma(w) \nabla_v - w(\gamma(v)) \\ &\quad - \gamma(v) \nabla(w) + \gamma(w) \gamma(v). \end{aligned}$$

and

$$\nabla_{[v,w]}^\gamma = \nabla_{[v,w]} - \gamma([v,w]).$$

Now, as $d\gamma = 0$, we have:

$$(9) \quad v(\gamma(w) - w(\gamma(v))) = \gamma([v,w]).$$

The conclusion follows. \square

Remark 3.3. As $\tilde{R}^\theta = R^{\frac{1}{2}\kappa + \theta}$, we also have $\tilde{R}^\theta = R$.

3.4. A twisted basic Weitzenböck formula. We present now the Weitzenböck-type formula for the Laplace operator $\tilde{\Delta}_{b,\theta}$, canonically constructed by using the derivative $\tilde{d}_{b,\theta}$ on $\Omega_b(\mathcal{F})$.

Note first that

$$\begin{aligned} (10) \quad \tilde{\Delta}_{b,\theta} &= \tilde{D}_{b,\theta}^2 \\ &= \sum_{i,j} (e_i \cdot \tilde{\nabla}_{e_i}^\theta)(e_j \cdot \tilde{\nabla}_{e_j}^\theta) - 2 \sum_i \iota_{\theta^\#}(e_i \cdot \tilde{\nabla}_{e_i}^\theta) \\ &\quad - 2 \sum_i (e_i \cdot \tilde{\nabla}_{e_i}^\theta) \iota_{\theta^\#} + 4 \iota_{\theta^\#} \iota_{\theta^\#}. \end{aligned}$$

As the last term vanishes, we compute below the other three terms.

For the first one, using Lemma 3.3, we get

$$\begin{aligned} \sum_{i,j} (e_i \cdot \tilde{\nabla}_{e_i}^\theta)(e_j \cdot \tilde{\nabla}_{e_j}^\theta) &= \sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \tilde{\nabla}_{e_j}^\theta \\ &\quad + \sum_{i,j} e_i \cdot e_j \cdot \tilde{\nabla}_{e_i}^\theta \cdot \tilde{\nabla}_{e_j}^\theta. \end{aligned}$$

Let Γ_{ij}^k be the Christoffel coefficients given by $\nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k$, for the local orthonormal frame $\{e_i\}_{1 \leq i \leq q}$. As e_i are basic, Γ_{ij}^k are basic functions. Then:

$$\begin{aligned}
\sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \tilde{\nabla}_{e_j}^\theta &= \sum_{i,j,k} e_i \cdot -\Gamma_{ik}^j e_k \cdot \tilde{\nabla}_{e_j}^\theta \\
&= -\sum_{i,k} e_i \cdot e_k \cdot \tilde{\nabla}_{\nabla_{e_i} e_k}^\theta,
\end{aligned}$$

and we obtain

$$\begin{aligned}
(11) \quad \sum_{i,j} (e_i \cdot \tilde{\nabla}_{e_i}^\theta)(e_j \cdot \tilde{\nabla}_{e_j}^\theta) &= \sum_i \tilde{\nabla}_{\nabla_{e_i} e_i}^\theta - \sum_i \tilde{\nabla}_{e_i}^\theta \tilde{\nabla}_{e_i}^\theta \\
&\quad + \sum_{i < j} e_i \cdot e_j \cdot (\tilde{\nabla}_{e_i}^\theta \tilde{\nabla}_{e_j}^\theta - \tilde{\nabla}_{e_j}^\theta \tilde{\nabla}_{e_i}^\theta - \tilde{\nabla}_{\nabla_{e_i} e_j - \nabla_{e_j} e_i}^\theta).
\end{aligned}$$

As

$$\sum_i \nabla_{e_i} e_i = -\sum_i g(\nabla_{e_i} e_j, e_i) e_j = -\sum_i \operatorname{div}^\nabla e_i e_i,$$

the ‘rough’ Laplace operator $(\tilde{\nabla}^\theta)^2$ of the basic twisted connection satisfies the equation:

$$\begin{aligned}
(\tilde{\nabla}^\theta)^2 &= \sum_i \tilde{\nabla}_{\nabla_{e_i} e_i}^\theta - \sum_i \tilde{\nabla}_{e_i}^\theta \tilde{\nabla}_{e_i}^\theta \\
&= \sum_i -\operatorname{div}^\nabla e_i \tilde{\nabla}_{e_i}^\theta - \sum_i \tilde{\nabla}_{e_i}^\theta \tilde{\nabla}_{e_i}^\theta \\
&= \sum_i \tilde{\nabla}_{e_i}^{\theta*} \tilde{\nabla}_{e_i}^\theta + 2 \sum_i \theta(e_i) \tilde{\nabla}_{e_i}^\theta \quad \text{by Lemma 3.2.}
\end{aligned}$$

Observe that, according to the definition of the twisted Bott connection, in (11) we do not have yet a basic curvature operator associated to $\tilde{\nabla}^\theta$. However, using Remark 3.1, we obtain

$$\tilde{\nabla}_{\pi_{T\mathcal{F}}([e_i, e_j])}^\theta \alpha = 0,$$

for any $\alpha \in \Omega_b(\mathcal{F})$. Then, using also Remark 3.3, we denote

$$\begin{aligned}
\mathcal{R} &:= \sum_{i < j} e_i \cdot e_j \cdot (\tilde{\nabla}_{e_i}^\theta \tilde{\nabla}_{e_j}^\theta - \tilde{\nabla}_{e_j}^\theta \tilde{\nabla}_{e_i}^\theta - \tilde{\nabla}_{\nabla_{e_i} e_j - \nabla_{e_j} e_i}^\theta) \\
&= \sum_{i < j} e_i \cdot e_j \cdot \tilde{R}_{e_i, e_j}^\theta \\
&= \sum_{i < j} e_i \cdot e_j \cdot R_{e_i, e_j}.
\end{aligned}$$

With this, we finally obtain

$$(12) \quad \sum_{i,j} (e_i \cdot \tilde{\nabla}_{e_i}^\theta)(e_j \cdot \tilde{\nabla}_{e_j}^\theta) = \sum_i \tilde{\nabla}_{e_i}^{\theta*} \tilde{\nabla}_{e_i}^\theta + 2 \sum_i \theta(e_i) \tilde{\nabla}_{e_i}^\theta + \mathcal{R}.$$

As for the second term in (10), using Lemma 3.5, we get

$$\begin{aligned}
(13) \quad \sum_i \iota_{\theta^\sharp}(e_i \cdot \tilde{\nabla}_{e_i}^\theta) &= \sum_i \iota_{\theta^\sharp} e^i \tilde{\nabla}_{e_i}^\theta - \sum_i e_i \cdot \iota_{\theta^\sharp} \tilde{\nabla}_{e_i}^\theta \\
&= \sum_i \theta(e_i) \tilde{\nabla}_{e_i}^\theta - \sum_i e_i \cdot \iota_{\theta^\sharp} \tilde{\nabla}_{e_i}^\theta.
\end{aligned}$$

For the third term we use Lemma 3.4:

$$(14) \quad \sum_i (e_i \cdot \tilde{\nabla}_{e_i}^\theta) \iota_{\theta^\sharp} = \sum_i e_i \cdot \iota_{\theta^\sharp} \tilde{\nabla}_{e_i}^\theta + \sum_i e_i \cdot \iota_{\nabla_{e_i} \theta^\sharp}.$$

Plugging equations (12)-(14) in the relation (10), we end up with the corresponding version of Weitzenböck formula for the Laplace-type operator $\tilde{\Delta}_{b,\theta}$:

$$(15) \quad \begin{aligned} \tilde{\Delta}_{b,\theta} &= \sum_i \tilde{\nabla}_{e_i}^{\theta*} \tilde{\nabla}_{e_i}^\theta + 2 \sum_i \theta(e_i) \tilde{\nabla}_{e_i}^\theta + \mathcal{R} - 2 \sum_i \theta(e_i) \tilde{\nabla}_{e_i}^\theta \\ &\quad + 2 \sum_i e_i \cdot \iota_{\theta^\sharp} \tilde{\nabla}_{e_i}^\theta - 2 \sum_i e_i \cdot \iota_{\theta^\sharp} \tilde{\nabla}_{e_i}^\theta - 2 \sum_i e_i \cdot \iota_{\nabla_{e_i} \theta^\sharp} \\ &= \sum_i \tilde{\nabla}_{e_i}^{\theta*} \tilde{\nabla}_{e_i}^\theta - 2 \sum_i e_i \cdot \iota_{\nabla_{e_i} \theta^\sharp} + \mathcal{R}. \end{aligned}$$

Remark 3.4. In general, it is difficult to obtain a convenient basic Weitzenböck formula. First of all, if $\theta = 0$, then we obtain the classical form of such formula for $\tilde{\Delta}_b$ (for the particular case when the mean curvature κ is also harmonic, see also [HR]). Secondly, as we noticed before, for the particular case when $\theta = -1/2 \cdot \kappa$, we obtain the classical basic Laplace operator, [RP]. Assuming that κ is not only basic, but also parallel with respect to the Bott connection (with the corresponding topological consequences), we obtain again a standard form for (15).

In the next section we work out the Bochner technique for the basic Morse-Novikov cohomology, deriving corresponding conditions that imply the vanishing of the cohomology groups.

4. VANISHING RESULTS FOR THE BASIC MORSE-NOVIKOV COHOMOLOGY

In this section we investigate several situations when the groups of basic Morse-Novikov cohomology vanish.

We start by extending to our context a result from [LLMP], where the authors prove that if the closed form θ is also parallel, then the Morse-Novikov cohomology becomes trivial.

We shall need the following commutation formulae, written using the operator $\tilde{\nabla}$. Recall that $\tilde{\nabla} = \nabla^{\frac{1}{2}\kappa} = \tilde{\nabla}^0$.

Lemma 4.1. *If the closed one-form θ is parallel with respect to the Bott connection, then the following equations are satisfied:*

$$[\tilde{\nabla}_{\theta^\sharp}, \tilde{d}_{b,\theta}] = 0, \quad [\tilde{\nabla}_{\theta^\sharp}, \tilde{\delta}_{b,\theta}] = 0.$$

Proof. We have

$$(16) \quad \begin{aligned} \tilde{\nabla}_{\theta^\sharp} &= \sum_i e^i \wedge \iota_{\theta^\sharp} \tilde{\nabla}_{e_i} + \sum_i \theta(e_i) \tilde{\nabla}_{e_i} - \sum_i e^i \wedge \iota_{\theta^\sharp} \tilde{\nabla}_{e_i} \\ &= \sum_i e^i \wedge \tilde{\nabla}_{e_i} \iota_{\theta^\sharp} + \iota_{\theta^\sharp} \sum_i e^i \wedge \tilde{\nabla}_{e_i} \\ &= \tilde{d}_b \iota_{\theta^\sharp} + \iota_{\theta^\sharp} \tilde{d}_b, \end{aligned}$$

where we use the fact that θ is parallel.

We achieve the result in two steps. Firstly, using (16) we show that $\tilde{d}_b, \tilde{\delta}_b$ commute with $\tilde{\nabla}_{\theta^\sharp}$.

$$(17) \quad \tilde{\nabla}_{\theta^\sharp} \tilde{d}_b = \tilde{d}_b \iota_{\theta^\sharp} \tilde{d}_b = \tilde{d}_b (\tilde{\nabla}_{\theta^\sharp} - \tilde{d}_b \iota_{\theta^\sharp}) = \tilde{d}_b \tilde{\nabla}_{\theta^\sharp}.$$

From Lemma 3.2, as θ is parallel with respect to the Bott connection, we derive that $\tilde{\nabla}_{\theta^\sharp}^* = -\tilde{\nabla}_{\theta^\sharp}$. Taking adjoint operators, we obtain:

$$\tilde{\nabla}_{\theta^\sharp} \tilde{\delta}_b = \tilde{\delta}_b \tilde{\nabla}_{\theta^\sharp}.$$

Using the fact that θ is parallel, the following Leibniz rule is easy to prove for the connection $\tilde{\nabla}$ and any $v \in \Gamma(Q)$.

$$\tilde{\nabla}_v \theta \wedge = (\nabla_v \theta) \wedge + \theta \wedge \tilde{\nabla}_v = \theta \wedge \tilde{\nabla}_v.$$

Then, for the twisted operators, this gives:

$$(18) \quad \begin{aligned} \tilde{\nabla}_{\theta^\sharp} \tilde{d}_{b,\theta} &= \tilde{\nabla}_{\theta^\sharp} \tilde{d}_b - \tilde{\nabla}_{\theta^\sharp} \theta \wedge \\ &= (\tilde{d}_b - \theta \wedge) \tilde{\nabla}_{\theta^\sharp} \\ &= \tilde{d}_{b,\theta} \tilde{\nabla}_{\theta^\sharp}. \end{aligned}$$

Again taking adjoint operators, we obtain

$$(19) \quad \tilde{\nabla}_{\theta^\sharp} \tilde{\delta}_{b,\theta} = \tilde{\delta}_{b,\theta} \tilde{\nabla}_{\theta^\sharp}.$$

□

The relation (16) allows us to establish the link between the operators $\tilde{d}_{b,\theta}$, ι_{θ^\sharp} and the twisted connection $\tilde{\nabla}_{\theta^\sharp}$, namely

$$(20) \quad \begin{aligned} \tilde{\nabla}_{\theta^\sharp} - \|\theta\|^2 \text{Id} &= (\tilde{d}_b - \theta \wedge) \iota_{\theta^\sharp} + \iota_{\theta^\sharp} (\tilde{d}_b - \theta \wedge) \\ &= \tilde{d}_{b,\theta} \iota_{\theta^\sharp} + \iota_{\theta^\sharp} \tilde{d}_{b,\theta}. \end{aligned}$$

Now we can prove:

Theorem 4.1. *Let (M, \mathcal{F}, g) be a Riemannian foliation with closed manifold M and basic mean curvature κ . If the basic, nontrivial one-form θ is parallel with respect to the Bott connection ∇ , then*

$$\tilde{H}_{b,\theta}^i(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^i(\mathcal{F}) = 0$$

for $0 \leq i \leq q$, where $H_{b,\frac{1}{2}\kappa+\theta}^i$ are the basic Morse-Novikov cohomology groups.

Proof. Assume that the basic, nontrivial one-form θ is parallel with respect to the Bott connection ∇ . Similar to [LLMP], without restricting the generality we can also assume that $\|\theta\| = 1$, otherwise considering the conformal transformation of the metric $g' := \|\theta\|^2 g$ we obtain the desired condition. Let $\alpha \in \mathcal{H}^p(\tilde{\Delta}_{b,\theta})$, where, as above, $\mathcal{H}^p(\tilde{\Delta}_{b,\theta}) = \text{Ker} \tilde{\Delta}_{b,\theta}|_{\Omega_b(\mathcal{F})}$. Then $\tilde{d}_{b,\theta} \alpha = 0$, $\tilde{\delta}_{b,\theta} \alpha = 0$, α being a harmonic form associated to $\tilde{\Delta}_{b,\theta}$. As θ is parallel with respect to the Bott connection, using again that $\tilde{\nabla}_{\theta^\sharp}^* = -\tilde{\nabla}_{\theta^\sharp}$, we get

$$\langle \tilde{\nabla}_{\theta^\sharp} \alpha, \alpha \rangle = \langle \alpha, -\tilde{\nabla}_{\theta^\sharp} \alpha \rangle,$$

so

$$(21) \quad \langle \tilde{\nabla}_{\theta^\sharp} \alpha, \alpha \rangle = 0.$$

Now, equation (20) implies:

$$\tilde{\nabla}_{\theta^\sharp} \alpha - \alpha = \tilde{d}_{b,\theta} (\iota_{\theta^\sharp} \alpha),$$

and $[\tilde{\nabla}_{\theta^\#}\alpha] \equiv [\alpha]$ *i.e.* $\tilde{\nabla}_{\theta^\#}\alpha$ and α lie in the same cohomology class of $H_{b, \frac{1}{2}\kappa+\theta}^p$. Using the commutation relations (18) and (19), we prove that $\tilde{\nabla}_{\theta^\#}\alpha$ is also a harmonic form. Indeed,

$$\begin{aligned} \tilde{d}_{b,\theta}\tilde{\nabla}_{\theta^\#}\alpha &= \tilde{\nabla}_{\theta^\#}\tilde{d}_{b,\theta}\alpha = 0, \\ \tilde{\delta}_{b,\theta}\tilde{\nabla}_{\theta^\#}\alpha &= \tilde{\nabla}_{\theta^\#}\tilde{\delta}_{b,\theta}\alpha = 0, \end{aligned}$$

and as a consequence we must have $\tilde{\nabla}_{\theta^\#}\alpha = \alpha$. From (21) it follows that $\alpha = 0$, and hence $\mathcal{H}^p(\tilde{\Delta}_{b,\theta}) \equiv 0$, which proves the result. \square

Example 4.1. We construct a Riemannian foliation endowed with a parallel basic one-form and apply the above result.

Consider a Hopf manifold constructed in the following manner (see *e.g.* [DO, V1]). On the complex manifold $\mathbb{C}^n \setminus \{0\}$ one considers the metric $g := z \mapsto 1/|z|^2 \cdot g_0$, where g_0 is the canonical Euclidean metric, with $z := (z^1, \dots, z^n)$. Let Δ be the cyclic group generated by the transformation $z \mapsto e^2 z$. The quotient $\mathbb{C}H := (\mathbb{C}^n \setminus \{0\})/\Delta$, is a *complex Hopf manifold*. Moreover, the above metric is invariant with respect to the transformation, and a quotient metric, still denoted by g , is induced on $\mathbb{C}H$. If $S^1(1/\pi)$ is the circle of radius $1/\pi$, then the mapping $f : \mathbb{C}^n \setminus \{0\} \rightarrow S^1(1/\pi) \times S^{2n-1}$ defined as

$$f(z) := \left(\frac{1}{\pi} e^{i\pi \ln|z|}, \frac{z}{|z|} \right)$$

is invariant with respect to the above group action, and induces an isometry between $\mathbb{C}H$ and $S^1(1/\pi) \times S^{2n-1}$ [DO]. If J is the complex structure, then the Kähler form $\omega(\cdot, \cdot) := g(\cdot, J\cdot)$ has the expression

$$\omega = -i \frac{1}{2|z|^2} \sum_j dz^j \wedge d\bar{z}^j.$$

One easily sees that ω satisfies the equation (see Section 5)

$$d\omega = \theta \wedge \omega, \quad \text{with} \quad \theta = -\frac{1}{|z|^2} \sum_j (z^j d\bar{z}^j + \bar{z}^j dz^j).$$

This θ is called *Lee form* and, in this case, it is parallel with respect to the Levi-Civita connection of g . Its metric dual, called the *Lee field*:

$$B := \theta^\# = -\frac{1}{2} \sum_j \left(z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} \right)$$

is also parallel, and consequently it is Killing, so we have (see *e.g.* [DO])

$$\mathcal{L}_B g = 0, \quad \mathcal{L}_B \theta = 0.$$

If φ is the flow generated by B and if $T \in \mathbb{R}$ is fixed, then φ_T is an isometry of $\mathbb{C}H$ which leaves g and θ invariant. Then, the direct product space $\hat{M} := \mathbb{C}H \times \mathbb{R}$ is endowed with a direct product Riemannian structure (on \mathbb{R} we just consider the standard metric). Moreover, \hat{M} is foliated by the real lines, and θ is in fact a basic one-form which is parallel with respect to the Bott connection. We can now *suspend* (see *e.g.* [Mo]) the action of φ_T on $\mathbb{C}H$ by considering the equivalence relation $(y, x) \sim (\varphi_T(y), x+1)$ on \hat{M} and taking the quotient

$$M := \hat{M} / \sim.$$

We end up with a Riemannian foliation (M, \mathcal{F}) on which the basic one-form θ remains parallel. Applying Theorem 4.1, the basic Morse-Novikov cohomology groups $\{\tilde{H}_{b,\theta}^i(\mathcal{F})\}_{0 \leq i \leq q}$ are trivial for the above suspension of the Hopf manifold.

In what follows we investigate the weaker case when θ is closed, but non-exact and non-parallel. We adapt the arguments from [GL] (see also [DO]) to our framework. More precisely, we prove the following statement:

Theorem 4.2. *Let (M, \mathcal{F}, g) be a transversally oriented Riemannian foliation with the underlying manifold M closed and connected, and suppose the mean curvature κ basic. If the basic one-form θ is closed but not exact, then the top dimension basic Morse-Novikov cohomology group $\tilde{H}_{b,\theta}^q(\mathcal{F})$ vanishes,*

$$\tilde{H}_{b,\theta}^q(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^q(\mathcal{F}) = 0.$$

Proof. Assume that $\alpha \in \Omega_b^q(\mathcal{F})$, $\alpha = f \cdot \text{vol}^Q$, where vol^Q is the (globally defined) transverse volume form. As α and vol^Q are basic (leafwise invariant) differential forms, then f will be a basic (*i.e.* constant along leaves) smooth function. Locally we may write:

$$(22) \quad \alpha = f e^1 \wedge \cdots \wedge e^q,$$

with respect to the locally defined basic orthonormal coframe $\{e^i\}_{1 \leq i \leq q}$. Clearly

$$\tilde{d}_{b,\theta+\frac{1}{2}\kappa}(\alpha) = 0,$$

and let us assume that also

$$\tilde{\delta}_{b,\theta+\frac{1}{2}\kappa}(\alpha) = 0.$$

Then we have

$$\begin{aligned} \tilde{\delta}_{b,\theta+\frac{1}{2}\kappa}(\alpha) &= - \sum_i e^i \wedge \tilde{\nabla}_{e_i} \alpha - \iota_{\theta+\frac{1}{2}\kappa} \alpha \\ &= - \sum_i e^i \wedge \nabla_{e_i} \alpha - \iota_{\theta} \alpha. \end{aligned}$$

We consider the local descriptions $\theta = \sum_i \theta_i e^i$ and also (22).

$$\tilde{\delta}_{b,\theta+\frac{1}{2}\kappa}(\alpha) = - \sum_i \iota_{e_i} \nabla_{e_i} (f e^1 \wedge \cdots \wedge e^q) - \iota_{\theta_i e_i} (f e^1 \wedge \cdots \wedge e^q).$$

From here, as vol^Q is parallel with respect to ∇ , we find

$$e_i(f) + f \theta_i = 0,$$

for $1 \leq i \leq q$. As f is basic, we obtain

$$d_b f + f \theta = \sum_i e^i \wedge \nabla_{e_i} f + \sum_i f \theta_i e_i = 0.$$

Assuming that the function f is nowhere vanishing, we can write

$$(23) \quad \theta = d(-\ln f),$$

which is a contradiction with the initial assumption. Then the zero set of f cannot be empty.

Consider a finite open cover $\{U_i\}_{i \in I}$ of M , such that (U_i, φ_i) are contractible foliated local maps. Then we can find some positive basic functions ψ_i satisfying the property

$$(24) \quad \theta|_{U_i} = -d(\ln \psi_i).$$

In fact we can construct such function in a canonical way on a local transversal T such that the above relation is fulfilled for the local projection of the basic (projectable) one-form θ on T ; then we can take the pull-back of the function on U_i , obtaining a local basic function. Then (23) and (24) imply $f = \mathbf{c}_i \psi_i$ on U_i , with $\mathbf{c}_i \in \mathbb{R}$. If $f = 0$ at some point then necessarily $\mathbf{c}_i = 0$, and f vanishes on a whole open neighborhood. So the zero set is open. Clearly the zero set of f is also closed, and hence it coincides with the connected manifold M , and $\alpha = 0$. Consequently there is no basic harmonic form of degree q with respect to $\tilde{\Delta}_{b,\theta}$. Now the Hodge decomposition theorem yields $\tilde{H}_{b,\theta}^i(\mathcal{F}) = H_{b,\frac{1}{2}\kappa+\theta}^q(\mathcal{F}) = 0$. \square

Remark 4.1. We point out that a theory of geometric objects which would be at the same time leafwise invariant and compactly supported is not easy to undertake, so the extension of the above result to the noncompact case is not trivial.

Example 4.2. We now present an application for the above theorem. A classical Riemannian flow can be constructed starting with a matrix $A \in \text{SL}(2, \mathbb{Z})$, with $\text{Tr} A > 2$ [Ca, Mo, T]. If $\{\lambda_i\}_{1 \leq i \leq 2}$ are the eigenvalues of A , it is easy to see that

$$\lambda_i \neq 1, \lambda_i > 0.$$

Let $\{v_i\}_{1 \leq i \leq 2}$ be the corresponding orthonormal eigenvectors. We denote by H the space \mathbb{R}^3 regarded as

$$(25) \quad \mathbb{R}^3 \equiv \mathbb{R} \times \mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R}v_1 \times \mathbb{R}v_2$$

We endow H with a Lie group structure using the multiplication

$$p \cdot p' := (t + t', \lambda_1^t \alpha' + \alpha, \lambda_2^t \beta' + \beta),$$

for any $p = (t, \alpha, \beta)$, $p' = (t', \alpha', \beta') \in H$, with respect to the identification (25) of \mathbb{R}^3 . Starting with the orthonormal basis $\{e := (1, 0, 0), v_1, v_2\}$, we construct three left invariant vector fields on H such that at any point p we have

$$\begin{aligned} e_p &= (1, 0, 0), \\ v_{1,p} &= \lambda_1^t (0, 1, 0), \\ v_{2,p} &= \lambda_2^t (0, 0, 1), \end{aligned}$$

which, in turn, generate the warped metric

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1^{-2t} & 0 \\ 0 & 0 & \lambda_2^{-2t} \end{pmatrix}.$$

The manner we choose the matrix A insures that the standard subgroup $\Gamma := \mathbb{Z} \times \mathbb{Z}^2$ of \mathbb{R}^3 remains a discrete and cocompact subgroup of H . Consequently, we obtain the quotient Lie group $T_A^3 := \Gamma \backslash H$. It also inherits the above left invariant geometric objects, which will be denoted in the same way, for convenience. The flow φ_2 generated by v_2 induces on our manifold a foliation structure \mathcal{F} , which can be proved to be a GA -foliated manifold, where GA is the the *orientation preserving affine group* [Ca].

We compute the Lie brackets

$$\begin{aligned} [e, v_2] &= \ln \lambda_1 v_2, \\ [e, v_3] &= \ln \lambda_2 v_3, \\ [v_2, v_3] &= 0, \end{aligned}$$

and we use the classical Koszul formula for the metric g on T_A^3 to obtain

$$\begin{aligned} g(\nabla_{v_2} v_2, e) &= \ln \lambda_2, \\ g(\nabla_{v_2} v_2, v_1) &= 0. \end{aligned}$$

Then $\kappa^\sharp = \ln \lambda_2 e$. As $e^\flat = dt$, we finally find

$$\kappa = \ln \lambda_2 dt.$$

It is now easy to see that on the compact quotient manifold T_A^3 the basic form dt is closed but not exact. Furthermore, $H_b^1(\mathcal{F}) \equiv \mathbb{R} [\text{Ca}]$, and the basic differential one-forms θ that are closed but not exact are precisely

$$\theta = c dt + df, \quad \text{with } c \in \mathbb{R} \setminus \{0\} \text{ and } f \in \Omega_b^0(\mathcal{F}).$$

Applying Theorem 4.2 for the above considered Riemannian flow, we obtain the vanishing of the top dimension group of the basic Morse-Novikov cohomology complex $(\Omega_b(\mathcal{F}), \tilde{d}_{b,\theta})$, *i.e.*

$$\tilde{H}_{b,\theta}^2(\mathcal{F}) = 0.$$

This result can be regarded as a generalization of [Ca, III, Proposition 2]. Indeed, for the particular choice $\theta = -1/2 \cdot \ln \lambda_2 dt$, we obtain the vanishing of the basic cohomology group $H_b^2(\mathcal{F})$, with the corresponding tautness consequences. For the particular case $\theta = 0$ see also [HR].

In the following, we investigate the vanishing of the basic Morse-Novikov cohomology under the assumption of certain conditions related to the curvature-type operators. Note that in the classical case represented by closed Riemannian manifolds, if the de Rham cohomology vanishes, then all closed differential one-forms are in fact exact. The Morse-Novikov cohomology complex, being isomorphic to de Rham complex (as we noticed in the Section 2.2), vanishes, too. If the curvature operator is non-negative and positive at some point, applying a well known result of Gallot and Meyer [GM] on closed Riemannian manifolds, we obtain that the Morse-Novikov cohomology complex is trivial in this case.

In the general case of a Riemannian foliation (not necessarily with basic mean curvature) for any 1-form θ which is basic and exact with respect to the operator d_b , we can consider a basic function f with $d_b f = \theta$; then, similar to the classical case we obtain an isomorphism between the basic de Rham and Morse-Novikov cohomologies using the mapping $[\alpha] \rightarrow [e^{-f} \alpha]$.

Now, in order to obtain the vanishing of the basic Morse-Novikov cohomology complex in our framework, the only needed ingredient is the corresponding version of the result of Gallot and Meyer. This was achieved in [He, MRT], where the authors used arguments related to functional analysis and operator theory.

Consequently, we easily obtain the following result:

Proposition 4.1. *If a Riemannian foliation (M, \mathcal{F}, g) has non-negative, and positive at some point, basic curvature operator, then any closed basic 1-form θ is exact, and consequently*

$$H_{b,\theta}^i(\mathcal{F}) = 0, \quad 0 < i < q.$$

Remark 4.2. We notice that, in accordance with Remark 3.4, for the particular case when κ is basic and parallel, the result from [He, MRT] can be derived in the classical fashion.

Example 4.3. As an application of Proposition 4.1, we consider the case of a suspension foliation used by Connes [Co] (see also [Mo, Appendix E]). More precisely, let S be a compact orientable surface S of genus 2, with universal cover \tilde{S} , and define $\tilde{M} := \mathrm{SO}(3, \mathbb{R}) \times \tilde{S}$. Let $h : \pi_1(S) \rightarrow \mathrm{SO}(3, \mathbb{R})$ be a group homomorphism. Define a smooth *diagonal* action of $\pi_1(S)$ on \tilde{M} by setting

$$R_{[\gamma]}(y, \hat{x}) = (h([\gamma]^{-1})(y), \hat{x} \circ [\gamma])$$

for each $[\gamma] \in \pi_1(S)$.

The quotient manifold $M := \tilde{M}/R$ is then a $\mathrm{SO}(3, \mathbb{R})$ -foliation. If, moreover, h is injective, then the leaves are actually diffeomorphic to \mathbb{R}^2 .

Endow \tilde{M} with a direct product Riemannian metric, which is also invariant with respect to the above action R . Then the foliated manifold M inherits a bundle-like metric. Concerning the transverse part, if the image of h is represented by canonical mappings produced by taking left multiplications on $\mathrm{SO}(3, \mathbb{R})$, then a left invariant metric g can be defined in a standard way on our Lie group. For instance, in the Lie algebra $\mathfrak{o}(3, \mathbb{R})$ we choose e_1, e_2 and e_3 as

$$e_1 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the metric is constructed such that $\{e_i\}_{1 \leq i \leq 3}$ becomes an orthonormal basis. As the group is compact, the metric is in fact bi-invariant, and the following formula can be used to compute the sectional curvature k (see *e.g.* [Mi]).

$$k(e_i, e_j) = \frac{1}{4} g([e_i, e_j], [e_i, e_j]), \quad 1 \leq i, j \leq 3.$$

As

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2,$$

we obtain

$$k(e_1, e_2) = k(e_2, e_3) = k(e_3, e_1) = \frac{1}{4},$$

$\mathrm{SO}(3, \mathbb{R})$ being in fact a distinguished compact Lie group admitting metrics with strictly positive curvature [Mi].

Proposition 4.1 now implies that the Morse-Novikov cohomology groups are trivial for the Connes foliation.

On the other hand, in [HR, Corollary 6.8], for the particular case when $\theta \equiv 0$, the authors obtain a vanishing result for the groups of basic Morse-Novikov cohomology which holds even in the case when the basic curvature operator is not necessarily non-negative. Note that in this case the basic de Rham complex may not be trivial. Starting from this remark, we obtain the following vanishing result for a closed basic one-form θ .

Theorem 4.3. *Let (M, \mathcal{F}, g) be a Riemannian foliation with M closed and the mean curvature κ basic. Assume that θ is a basic closed one-basic form and define the bi-linear map*

$$(26) \quad \beta_\theta : \Omega_b(\mathcal{F}) \times \Omega_b(\mathcal{F}) \rightarrow C^\infty(M), \text{ with } \beta_\theta(\cdot, \cdot) := \mathcal{L}_{\theta^\sharp} g(\cdot, \cdot) + g(\mathcal{R}\cdot, \cdot).$$

If β is non-negatively defined, and β_x is positively defined at some point $x \in M$, then the basic Morse-Novikov cohomology groups vanish

$$\tilde{H}_{b,\theta}^i(\mathcal{F}) = 0, \quad 0 < i < q.$$

Proof. We start from (15) and let $\alpha \in \Omega^i(\mathcal{F})$, $0 < i < q$. Taking integrals on the closed manifold M , we obtain

$$\langle \tilde{\Delta}_{b,\theta} \alpha, \alpha \rangle = \sum_i \left\| \tilde{\nabla}_{e_i}^\theta \alpha \right\|^2 - 2 \int_M g\left(\sum_i e_i \cdot \iota_{\nabla_{e_i} \theta^\sharp} \alpha, \alpha\right) d\mu_g + \int_M g(\mathcal{R}\alpha, \alpha) d\mu_g.$$

where, for arbitrary $\alpha \in \Omega^i(\mathcal{F})$, we define $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. Then, in order to apply the Bochner technique, we take a closer look at the middle term. As the scalar product of forms of different degrees vanishes, Lemma 3.5 implies:

$$\begin{aligned} g\left(\sum_i e_i \cdot \iota_{\nabla_{e_i} \theta^\sharp} \alpha, \alpha\right) &= g\left(\sum_i e^i \wedge \iota_{\nabla_{e_i} \theta^\sharp} \alpha, \alpha\right) \\ &= g(\mathcal{L}_{\theta^\sharp} \alpha, \alpha) - g(\nabla_{\theta^\sharp} \alpha, \alpha) \\ &= -\frac{1}{2} \mathcal{L}_{\theta^\sharp} g(\alpha, \alpha). \end{aligned}$$

This yields the formula

$$\langle \tilde{\Delta}_{b,\theta} \alpha, \alpha \rangle = \sum_i \left\| \tilde{\nabla}_{e_i}^\theta \alpha \right\|^2 + \int_M (\mathcal{L}_{\theta^\sharp} g(\alpha, \alpha) + g(\mathcal{R}\alpha, \alpha)) d\mu_g.$$

Arguing now as in the classical case, and considering the isomorphism $\mathcal{H}^i(\tilde{\Delta}_{b,\theta}) \simeq H_{b,\frac{1}{2}\kappa+\theta}^i(\mathcal{F})$, we obtain the result. \square

We outline the particular case of Theorem 4.3 obtained for $\theta = -1/2 \cdot \kappa$. Then, the basic Morse-Novikov cohomology complex is again just the classical basic de Rham complex, and we get vanishing conditions suitable for Riemannian foliations with non-positive transverse curvature which are different from the previous results (see *e.g.* [He, MRT]).

Corollary 4.1. *If the bi-linear map $\beta_{-\frac{1}{2}\kappa}$ defined in (26) is non-negatively defined and positively defined at some point $x \in M$, then the basic cohomology groups are trivial,*

$$H_b^i(\mathcal{F}) = 0, \quad 0 < i < q.$$

Furthermore, as $H_b^1(\mathcal{F}) \equiv 0$, the foliation is taut, [A].

Finally, another particular case is represented by closed Riemannian manifolds. Again, this classical framework is obtained for the limiting case when the leaves are points. The mean curvature vanishes, the basic geometric objects become the classical ones, and we obtain the Bochner technique adapted for Morse-Novikov cohomology.

Corollary 4.2. *Let (M, g) be a closed Riemannian manifold of dimension n and let θ be a closed differential one-form. If the bi-linear map*

$$\beta_\theta : \Omega(M) \times \Omega(M) \rightarrow \mathcal{C}^\infty(M), \text{ with } \beta_\theta(\cdot, \cdot) := \mathcal{L}_{\theta^\sharp} g(\cdot, \cdot) + g(\mathcal{R}\cdot, \cdot),$$

is non-negatively defined, and positively defined at some point $x \in M$, then Morse-Novikov cohomology groups vanish:

$$H_\theta^i(M) = 0, \text{ for } 0 < i < n.$$

5. APPLICATIONS TO L.C.S. AND L.C.K. FOLIATIONS

In this final section we apply the results obtained in the rest of the paper to the particular case represented by l.c.s. and l.c.K. foliations.

A *locally conformally symplectic* manifold (l.c.s.) is a differentiable manifold M of dimension $2n$ endowed with a differentiable form ω of dimension 2 which is locally conformal with a symplectic form (*i.e.* closed and non-degenerate differentiable form of dimension 2) [V3]: $d(e_U^f \cdot \omega|_U) = 0$. ω will be also called *locally conformally symplectic structure*. Following [DO, V1], we make the convention that the case when this procedure can be performed globally is not viewed as a particular case of l.c.s., but as an opposite case.

A condition equivalent to the definition is the existence of a global closed one-form θ (called *Lee form*) such that

$$(27) \quad d_\theta \omega := d\omega - \theta \wedge \omega = 0.$$

Furthermore, assume the manifold to be complex and endowed with a metric g compatible with the complex structure J . Then, if ω is determined by J and g (*i.e.* $\omega(\cdot, \cdot) := g(\cdot, J\cdot)$), then the manifold is said to be *locally conformally Kähler*, in this latter case a local Kähler metric being obtained by a conformal change of the initial metric. If the Lee form θ is parallel with respect to the Levi-Civita connection, then the manifold M is a *Vaisman manifold* (previously called *generalized Hopf manifold* [DO, V2]).

Regarding (27), we see that these types of differentiable manifolds have a natural Morse-Novikov cohomological complex (called also *adapted cohomology*), which encodes many interesting properties of the underlying manifolds (see [OV2, V2]).

The above defined geometric structures can be extended to the context of Riemannian foliations, the transverse geometry of the foliations corresponding to the geometry of the manifolds. We thus obtain *l.c.s. foliations*, *Kähler* and *l.c.K. foliations* and, respectively, *Vaisman foliations* (see [BD, IP]). We emphasize the interplay between the above defined spaces: for instance, classical Vaisman manifolds are examples of Kähler foliations of dimension 2 [BD].

The simplest examples of l.c.s. and l.c.K. foliations are represented by a smooth Riemannian submersion $f : M \rightarrow N$, the base manifold N being a l.c.s. (l.c.K., respectively) manifold [BD]. In turn, Proposition 4.1 provides a condition for the non-existence of such transverse structures.

Corollary 5.1. *Let (M, \mathcal{F}) be a foliated manifold of codimension $2q$, with M compact. Assume there is a bundle-like metric g on M such that the basic curvature operator is non-negatively defined and positively defined at some point. Then the foliation does not admit a transverse locally symplectic structure (and consequently there is no transverse l.c.K. structure with respect to any bundle-like metric defined on (M, \mathcal{F})).*

The proof is straightforward, applying our previous results.

Now, assume that the basic curvature operator allows the existence of a transverse l.c.s. structure ω with the transverse Lee form θ . Then, Theorem 4.3 provides vanishing conditions for the basic adapted cohomology.

Corollary 5.2. *If the bi-linear form β_θ defined in (26) is non-negatively defined and positively defined at some point, then the groups of the adapted basic cohomology vanish*

$$\tilde{H}_{b,\theta}^i = 0 \text{ for } 0 < i < 2q.$$

Note that the top dimension cohomology group cannot be addressed with the above result. In turn, this can be done using Theorem 4.2.

Corollary 5.3. *If the basic Lee form θ is not exact, then $\tilde{H}_{b,\theta}^{2q} = 0$.*

Remark 5.1. The above result stands as a generalization of [DO, Theorem 2.9] for the case when the manifold M is compact.

In the final part of the paper we deal with Vaisman foliations. We first observe that the suspension foliation constructed in Example 4.1, endowed with a parallel Lee form is consequently a Vaisman foliation. On the other hand, Theorem 4.1 implies that the groups of the basic adapted cohomology are trivial.

Corollary 5.4. *For any Vaisman foliation:*

$$\tilde{H}_{b,\theta}^i = 0, \text{ for } 0 < i < 2q.$$

Remark 5.2. The above result is an extension of [OV2], where the triviality of the adapted cohomology is derived directly from the structure theorem of compact Vaisman manifold [OV1]. For the general case of Riemannian foliations a similar attempt is not a trivial extension, as structural aspects of Riemannian foliations should be also considered [Mo].

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LIVIU ORNEA

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS,
14 ACADEMIEI STR., 70109 BUCHAREST, ROMANIA. *and*
INSTITUTE OF MATHEMATICS "SIMION STOILOW" OF THE ROMANIAN ACADEMY,
21, CALEA GRIVITEI STR. 010702-BUCHAREST, ROMANIA
lornea@fmi.unibuc.ro, liviu.ornea@imar.ro

VLADIMIR SLESAR

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRAIOVA,
13 AL.I. CUZA STR., 200585-CRAIOVA, ROMANIA
vlslesar@central.ucv.ro